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Normal pure states of the von Neumann algebra of bounded operators as Kähler manifold

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Abstract. The projective space of a complex Hilbert space $\mathscr{H}$ is considered both as a Kähler manifold and as the set of pure states of the von Neumann algebra $\mathscr{B}(\mathscr{H})$. A link is given between these two structures. Special attention is devoted to topology, orientation and automorphisms of the structures and Wigner's theorem.

1. Introduction

The set of pure states of ordinary quantum mechanics is the projective space $\mathbb{P}(\mathscr{H})$ of the Hilbert space $\mathscr{H}$ associated to the system. It is known that $\mathbb{P}(\mathscr{H})$ can be given a natural Kählerian structure (see e.g. Cirelli and Lanzavecchia 1981), from which the Schrödinger equation follows in a standard way. Now, pure states of ordinary quantum mechanics can be viewed either as normal pure states of the von Neumann algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on $\mathscr{H}$ or as pure states of the C^*-algebra $\mathscr{C}(\mathscr{H})$ of compact operators on $\mathscr{H}$.

Recently, Shultz (1980, 1982) has showed that the set of pure states of a C^*-algebra, as a uniform space equipped with transition probabilities and orientation, is a dual object for C^*-algebras, and determines the C^*-algebra up to *-isomorphisms. In Shultz's approach no differential structure is considered (only the trivial case of finite-dimensional C^*-algebras is mentioned); however, the interplay between differential geometrical and algebraic structures in the infinite-dimensional case seems to be quite interesting for foundations of quantum theories.

In this paper, we start this study with the simplest case of the duality between $\mathbb{P}(\mathscr{H})$ and von Neumann algebra $\mathscr{B}(\mathscr{H})$. In this duality $\mathbb{P}(\mathscr{H})$ corresponds to the boundary of the state space of $\mathscr{C}(\mathscr{H})$, and therefore inherits part of the affine structure of this convex set. With a slight abuse of language we refer to this structure as the affine structure of $\mathbb{P}(\mathscr{H})$.

In § 2 we briefly introduce the differential structure of $\mathbb{P}(\mathscr{H})$ and show that the underlying topology coincides with the (relativised) w^*-topology.

Section 3 is devoted to the study of Shultz's notion of orientation; we interpret orientation from the geometrical point of view, introducing the important notion of conjugation for $\mathbb{P}(\mathscr{H})$.

In § 4 we consider the energy function (related to the usual Schrödinger equation) and obtain it naturally from the algebraic duality relation between $\mathbb{P}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$.

The last section is devoted to the bijection between the automorphism group of $\mathscr{B}(\mathscr{H})$ and the group of Kähler isomorphisms of $\mathbb{P}(\mathscr{H})$. A new formulation of Wigner's theorem is obtained.

2. The topology on the projective space

The projective space $\mathbb{P}(\mathscr{H})$ of a complex Hilbert space $\mathscr{H}$ admits a natural structure of Kähler manifold (with Kähler metric $\mathscr{K}$). This structure is fully described in Cirelli and Lanzavecchia (1981), and here we want only to define it in a quicker way, without the use of local coordinates.

We denote by $[h]$ the subspace of $\mathscr{H}$ generated by $h \in \mathscr{H}-\{0\}$; define

$$V_h := \{[k] \in \mathbb{P}(\mathscr{H}), (h\,|\,k) \neq 0\}, \tag{1}$$

$$\varphi_h V_h \to \{h\}^{\perp}, \qquad \varphi_h([k]) := (h|k)^{-1} \cdot k - h. \tag{2}$$

The collection $\{(V_h, \varphi_h, \{h\}^{\perp})\}$ $(h \in \mathscr{H})$ is a holomorphic atlas $\mathscr{A}$ for $\mathbb{P}(\mathscr{H})$. $\mathscr{A}$ gives to $\mathbb{P}(\mathscr{H})$ a natural topology $\mathscr{T}_{\mathscr{K}}$.

Identifying every $[h] \in \mathbb{P}(\mathscr{H})$ with the corresponding one-dimensional projection, $\mathbb{P}(\mathscr{H})$ becomes the boundary of the positive part of the unit ball of the Banach space $\mathscr{T}\mathscr{C}(\mathscr{H})$ of trace class operators. It becomes therefore both the set of pure states of $\mathscr{C}(\mathscr{H})$ and the set of normal pure states of $\mathscr{B}(\mathscr{H})$. We now show that $\mathscr{T}_{\mathscr{K}}$ coincides with the relativised w^*-topology on $\mathbb{P}(\mathscr{H})$.

To do that, we have just to establish two facts:

(i) the maps φ_h $(h \in \mathscr{H})$ are w^*-continuous;

(ii) the maps $\tilde{C}$ $(C \in \mathscr{C}(\mathscr{H}))$ are $\mathscr{T}_{\mathscr{K}}$ continuous, where $\tilde{C}: \mathbb{P}(\mathscr{H}) \to \mathbb{C}$, $\tilde{C}(P) := \mathrm{Tr}(PC)$.

The statement (ii) may be replaced by

(ii′) the maps $\tilde{P}$ $(P \in \mathbb{P}(\mathscr{H}))$ are $\mathscr{T}_{\mathscr{K}}$ continuous.

Actually, though the weak topology induced by the maps $\tilde{C}$ with C of finite rank does not coincide with the w^*-topology on the whole $\mathscr{T}\mathscr{C}(\mathscr{H})$, it does on the unit ball (see Strătilă and Zsidó 1979, lemma 1.2), and therefore on $\mathbb{P}(\mathscr{H})$.

Let h, $k \in \mathscr{H}$, $\|h\| = \|k\| = 1$, $[k] \in V_h$, and let P, Q be the corresponding projections. As $\tilde{P}(Q) = |(h|k)|^2 \neq 0$, a trivial computation gives

$$\|\varphi_h([k])\|^2 = (1 - \tilde{P}(Q)) \cdot (\tilde{P}(Q))^{-1}, \tag{3}$$

and therefore

$$\tilde{P}(Q) = (1 + \|\varphi_h([k])\|^2)^{-1}. \tag{4}$$

From (3) we see that the norm of φ_h is w^*-continuous, and therefore φ_h is continuous in $[h]$, as $\varphi_h([h]) = 0$. Since φ_k is continuous in $[k]$, $\varphi_h = (\varphi_h \circ \varphi_k^{-1}) \circ \varphi_k$ and $\varphi_h \circ \varphi_k^{-1}$ is a biholomorphism, the continuity of φ_h in $[k]$ follows, and (i) is proved. From (4) we see that $\tilde{P}$ is $\mathscr{T}_{\mathscr{K}}$ continuous on V_h. As its extension to the whole $\mathbb{P}(\mathscr{H})$(given by $\tilde{P}(Q) = 0$ if $[k] \notin V_h$) is $\mathscr{T}_{\mathscr{K}}$ continuous too, we have proved (ii′).

We can therefore conclude with the following:

Theorem 1. The topology on $\mathbb{P}(\mathscr{H})$ underlying its Kähler structure equals the w^*-topology.

3. Geometrical interpretation of affine orientation

In this section we exploit the link established by theorem 1 between the Kählerian and affine structures on $\mathbb{P}(\mathscr{H})$. We recall the notion of orientation for pure states of a C^*-algebra A given in Shultz (1980, 1982), only specialising it to our case, where A is $\mathscr{C}(\mathscr{H})$. This notion admits a very simple characterisation in terms of the holomorphic atlas $\mathscr{A}$.

We denote by $\mathscr{H}(h, k)$ the subspace of $\mathscr{H}$ generated by $h, k \in \mathscr{H}-\{0\}$. The face of the state space of $\mathscr{C}(\mathscr{H})$ generated by $[h]$, $[k]$ is affinely isomorphic to the unit ball E^3 in $\mathbb{R}^3$, and its boundary $S^2([h], [k])$ is homeomorphic to S^2. The boundary of the face generated by $[h]$, $[k]$ is in fact the one-dimensional complex manifold $\mathbb{P}(\mathscr{H}(h, k))$, which is isomorphic to the complex projective space $\mathbb{P}(\mathbb{C}^2)$. This one, considered as a real manifold, is exactly the sphere S^2 with the canonical atlas. Moreover, there is a homeomorphism $\Xi: S^2 \to P(M_2(\mathbb{C}))$, where $P(M_2(\mathbb{C}))$ is the set of pure states of the C^*-algebra of 2×2 complex matrices. In the same way, if we consider $\mathbb{P}(\mathscr{H}(h, k))$ as a real manifold, we obtain $S^2([h], [k])$.

Let $\Xi_i: S^2 \to S^2([h], [k])$, $i=1, 2$, be bijections which preserve transition probabilities. Then $\Xi_2^{-1} \circ \Xi_1$ preserves transition probabilities, and extends uniquely to an orthogonal transformation of $\mathbb{R}^3$; we say Ξ_1 and Ξ_2 are equivalent if this transformation has determinant $+1$, and we refer to an equivalence class as an orientation of $S^2([h], [k])$.

Let P_{hk} be the projection on $\mathscr{H}(h, k)$. The C^*-algebra $A = P_{hk}\mathscr{C}(\mathscr{H})P_{hk}$ is $*$-isomorphic to $M_2(\mathbb{C})$, and its set of pure states is $S^2([h], [k])$. Given a $*$-isomorphism $\Phi: A \to M_2(\mathbb{C})$ we have a homeomorphism $\Phi^*: P(M_2(\mathbb{C})) \to S^2([h], [k])$. The orientation of $S^2([h], [k])$ given by $\Phi^* \circ \Xi$ does not depend on Φ, and is called the canonical orientation. The collection of the canonical orientations of every $S^2([h], [k])$ is the canonical orientation of $\mathbb{P}(\mathscr{H})$. More generally, an orientation of $\mathbb{P}(\mathscr{H})$ is a continuous assignment of an orientation on each $S^2([h], [k])$, that is, a continuous section of a suitable Z_2-bundle (see Alfsen *et al* 1980). $\mathbb{P}(\mathscr{H})$ can be given just the canonical orientation and the opposite one.

When one deals with geometrical and affine structure together, orientation becomes a misleading term; for this reason we call Shultz's orientation affine orientation. To connect affine orientation to differential geometry we make a short digression on geometrical orientation, with special regard to complex manifolds; we refer essentially to Flaherty (1976).

An m-dimensional complex manifold M with a holomorphic atlas $\mathscr{A}$ admits canonically a structure of real manifold, induced by a real atlas $\mathscr{A}^{\mathbb{R}}$, which is oriented. On the set M can be defined another complex atlas $\bar{\mathscr{A}}$, which is the conjugate of $\mathscr{A}$, and gives to M the conjugate differential structure. $\bar{\mathscr{A}}$ is called the antiholomorphic atlas for M. If m is even, the atlases $\mathscr{A}$ and $\bar{\mathscr{A}}$ induce the same geometrical orientation on the real manifold M. If m is odd (this is the case of $\mathbb{P}(\mathbb{C}^2) \simeq S^2$) the two atlases induce opposite geometrical orientations.

This geometrical notion of orientation is obviously linked to the finite dimension of M. We cannot therefore define such an orientation on $\mathbb{P}(\mathscr{H})$. The notion of affine orientation does not correspond to the geometrical notion; we will see that affine orientation corresponds to conjugation.

Let $h, k \in \mathscr{H}-\{0\}$. The restrictions of φ_h, φ_k to $V_h \cap \mathbb{P}(\mathscr{H}(h, k))$, $V_k \cap \mathbb{P}(\mathscr{H}(h, k))$ are a holomorphic atlas $\mathscr{A}_{hk}$ on $\mathbb{P}(\mathscr{H}(h, k))$. This atlas induces an oriented atlas on the real underlying manifold $S^2([h], [k])$, and therefore gives a geometrical orientation on it; thus, the holomorphic atlas $\mathscr{A}$ on $\mathbb{P}(\mathscr{H})$ induces a geometrical orientation on each $S^2([h], [k])$. The opposite geometrical orientation on $S^2([h], [k])$ is the one induced by the antiholomorphic atlas on $\mathbb{P}(\mathscr{H}(h, k))$. The collection of all the opposite orientations is therefore induced by the antiholomorphic atlas $\bar{\mathscr{A}}$ on $\mathbb{P}(\mathscr{H})$.

Let $\bar{\mathscr{H}}$ be the Hilbert space complex conjugate of $\mathscr{H}$. The set $\mathbb{P}(\bar{\mathscr{H}})$ is the set of normal pure states of $\mathscr{B}(\bar{\mathscr{H}})$, and equals $\mathbb{P}(\mathscr{H})$. We remark that $\mathscr{B}(\bar{\mathscr{H}})$ coincides with the opposite von Neumann algebra $\overline{\mathscr{B}(\mathscr{H})}$ of $\mathscr{B}(\mathscr{H})$. If we construct the holomorphic atlas $\mathscr{A}_{(\bar{\mathscr{H}})}$ on $\mathbb{P}(\bar{\mathscr{H}})$ (that is, the atlas which is obtained starting with $\bar{\mathscr{H}}$ instead of $\mathscr{H}$) we obtain the antiholomorphic atlas for $\mathbb{P}(\mathscr{H})$, which is now conveniently denoted by $\bar{\mathscr{A}}_{(\mathscr{H})}$. We have

$$(\mathbb{P}(\mathscr{H}), \bar{\mathscr{A}}_{(\mathscr{H})}) = (\mathbb{P}(\bar{\mathscr{H}}), \mathscr{A}_{(\bar{\mathscr{H}})}). \tag{5}$$

The atlas $\mathscr{A}_{(\mathscr{H})}$ induces on each $S^2([h], [k])$ a geometrical orientation. The opposite one, which is induced by the atlas $\bar{\mathscr{A}}_{(\mathscr{H})}$, can therefore be considered as the orientation induced by the atlas $\mathscr{A}_{(\bar{\mathscr{H}})}$ on $\mathbb{P}(\bar{\mathscr{H}})$.

We now define the notion of conjugation for $\mathbb{P}(\mathscr{H})$; then we will show that conjugation is the same thing as affine orientation. Consider the atlas on $\mathbb{P}(\mathscr{H})$ obtained by the union of $\mathscr{A}_{(\mathscr{H})}$ and $\bar{\mathscr{A}}_{(\mathscr{H})}$. This atlas is not holomorphic, and its charts can be divided into two equivalence classes, requiring the functions $\varphi_h \circ \varphi_k^{-1}$ to be holomorphic. We say that such an equivalence class is a conjugation for $\mathbb{P}(\mathscr{H})$. The canonical conjugation is the one induced by the atlas $\mathscr{A}_{(\mathscr{H})}$. By (5), the opposite conjugation is the canonical one for $\mathbb{P}(\bar{\mathscr{H}})$. Conjugation therefore allows us to distinguish if we are dealing with the set $\mathbb{P}(\mathscr{H}) = \mathbb{P}(\bar{\mathscr{H}})$ as the set of normal pure states of $\mathscr{B}(\mathscr{H})$ or of $\mathscr{B}(\bar{\mathscr{H}})$. These algebras have the same spaces of normal pure states, the same structure of Jordan algebras but not the same C^*-algebraic structure.

To discuss the relation between conjugation and affine orientation for $\mathbb{P}(\mathscr{H})$, let us consider again a $S^2([h], [k])$, and two transition probabilities preserving bijections $\Xi_i : S^2 \to S^2([h], [k])$, $i = 1, 2$. If Ξ_1 and Ξ_2 are equivalent, then $\Xi_2 \circ \Xi_1^{-1}$ is a bijection which preserves geometrical orientation. Actually, given a chart (V, γ) of the real oriented complete atlas $\mathscr{A}_{hk}^{\mathbb{R},C}$ induced by $\mathscr{A}_{hk}$, then $(\Xi_1 \circ \Xi_2^{-1}(V), \gamma \circ \Xi_2 \circ \Xi_1^{-1})$ belongs to $\mathscr{A}_{hk}^{\mathbb{R},C}$, that is the Jacobian determinant of $\gamma \circ (\gamma \circ \Xi_2 \circ \Xi_1^{-1})^{-1}$ is positive. If we start with two inequivalent transition probabilities preserving bijections, we obtain a bijection which reverses geometrical orientation, since the determinant of $\Xi_2^{-1} \circ \Xi_1$ equals -1.

Therefore, on each $S^2([h], [k])$, affine orientation coincides with geometrical orientation, since each equivalence class of transition probabilities preserving bijections corresponds to one of the geometrical orientations of the real manifold $S^2([h], [k])$. Moreover, the choice of a conjugation for $\mathbb{P}(\mathscr{H})$ induces a geometrical orientation (and therefore an affine orientation) on each $S^2([h], [k])$; we now show that the canonical conjugation and the canonical affine orientation for $\mathbb{P}(\mathscr{H})$ induce the same geometrical orientation (exactly the one given by $\mathscr{A}_{hk}$) on each $S^2([h], [k])$.

When we defined the atlas $\mathscr{A}_{hk}$ on $\mathbb{P}(\mathscr{H}(h, k))$ we used the atlas $\mathscr{A}$ on $\mathbb{P}(\mathscr{H})$. Another way to introduce $\mathscr{A}_{hk}$ is to consider any isomorphism $\Gamma : \mathscr{H}(h, k) \to \mathbb{C}^2$. From Γ we obtain a bijection $\tilde{\Gamma} : \mathbb{P}(\mathscr{H}(h, k)) \to \mathbb{P}(\mathbb{C}^2)$ which can be used to carry on $\mathbb{P}(\mathscr{H}(h, k))$ the canonical holomorphic atlas of $\mathbb{P}(\mathbb{C}^2)$. The resulting oriented atlas is equivalent to $\mathscr{A}_{hk}$. From Γ we can also obtain a $*$-isomorphism $\Phi_\Gamma : A \to M_2(\mathbb{C})$, where A is the C^*-algebra of linear operators on $\mathscr{H}(h, k)$, that is the above defined algebra $P_{hk}\mathscr{C}(\mathscr{H})P_{hk}$. It is now immediate that $\phi_\Gamma^* \circ \Xi : S^2 \simeq \mathbb{P}(\mathbb{C}^2) \to S^2([h], [k]) \simeq \mathbb{P}(\mathscr{H}(h, k))$ is exactly $\tilde{\Gamma}^{-1}$ (for the form of Ξ we again address the reader to Shultz (1980), where

Ξ is denoted by ψ). This means that $\phi_\Gamma^* \circ \Xi$ induces on $\overline{S^2([h], [k])}$ the geometrical orientation given by $\mathscr{A}_{hk}$. If we start with the algebra $\overline{\mathscr{B}(\mathscr{H})} \simeq \mathscr{B}(\bar{\mathscr{H}})$, the isomorphism $\bar{\Gamma}: \bar{\mathscr{H}}(h, k) \to \mathbb{C}^2$ induces on $\mathbb{P}(\bar{\mathscr{H}}(h, k))$ the atlas $\bar{\mathscr{A}}_{hk}$ and therefore the opposite geometrical and affine orientation.

Since the collection of affine orientations on each $S^2([h], [k])$ induced by a conjugation results in an affine orientation for $\mathbb{P}(\mathscr{H})$, we have the following:

Theorem 2. The canonical (resp. opposite) conjugation for $\mathbb{P}(\mathscr{H})$ corresponds exactly to the canonical (resp. opposite) affine orientation for $\mathbb{P}(\mathscr{H})$.

In our mind, conjugation for $\mathbb{P}(\mathscr{H})$ is defined in a simpler way than affine orientation for $\mathbb{P}(\mathscr{H})$; moreover, it points directly towards the reason for which affine orientation has been introduced, that is to have an object which allows us to distinguish between the sets $\mathbb{P}(\mathscr{H})$ and $\mathbb{P}(\bar{\mathscr{H}})$, and therefore between the two opposite algebras $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}(\bar{\mathscr{H}})$.

4. Energy function on $\mathbb{P}(\mathscr{H})$

The Kähler manifold $\mathbb{P}(\mathscr{H})$ has a canonical symplectic sructure. This allows us to associate to each (real) smooth function on $\mathbb{P}(\mathscr{H})$ a (real) vector field on $\mathbb{P}(\mathscr{H})$, that is an ordinary differential equation. Therefore, given a function (which we interpret as energy function of the physical system) we obtain in a natural way the motion equation (see Cirelli and Lanzavecchia 1981). This equation is equivalent to the usual Schrödinger equation (and is therefore physically correct) if the energy function is

$$\hbar_H([\psi]) = (\psi|H\psi) \cdot (\psi|\psi)^{-1}, \qquad \psi \in \mathscr{H} - \{0\}, \tag{6}$$

where H is a self-adjoint operator (the Hamiltonian operator) on $\mathscr{H}$. For the moment we assume H to be bounded. The self-adjointness of H is a necessary and sufficient condition for $\hbar_H$ to be real.

The form of the energy function (6), which was postulated in Cirelli and Lanzavecchia (1981), comes naturally from the duality relation between $\mathscr{T}\mathscr{C}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$: considering again $\mathbb{P}(\mathscr{H}) \subseteq \mathscr{T}\mathscr{C}(\mathscr{H})$, we can associate to each $H \in \mathscr{B}(\mathscr{H})_{sa}$ (self-adjoint part of $\mathscr{B}(\mathscr{H})$) a function $\tilde{H}: \mathbb{P}(\mathscr{H}) \to \mathbb{R}, \tilde{H}(P) = \mathrm{Tr}(PH)$. Of course, $\tilde{H}(P_{[\psi]}) = \hbar_H([\psi])$, where $P_{[\psi]}$ is the projection on $[\psi]$. The energy of a pure state $[\psi]$ is therefore the value on it of the linear functional associated to the Hamiltonian operator of the system.

Unbounded Hamiltonian operators can be treated with certain limit processes. We do not discuss this point here (see Cirelli and Lanzavecchia 1982).

5. Automorphism group of $\mathscr{B}(\mathscr{H})$, Kähler isomorphisms and Wigner's theorem

The automorphism group Aut $(\mathscr{B}(\mathscr{H}))$ of $\mathscr{B}(\mathscr{H})$ coincides with the group $\mathscr{U}(\mathscr{H})/\sim$, where $\mathscr{U}(\mathscr{H})$ is the group of unitary operators on $\mathscr{H}$, and $\sim$ is the equivalence relation defined by $U \sim U' \Longleftrightarrow \exists a \in \mathbb{R}: U = \exp(\mathrm{i}a)U'$.

Now we want to characterise the set of diffeomorphisms of $\mathbb{P}(\mathscr{H})$ which leave invariant the Kähler structure. For brevity, we simply discuss the group of automorphisms. This characterisation parallels closely that of symplectic diffeomorphisms (canonical transformations) in classical mechanics. A symplectic diffeomorphism of the phase space of classical mechanics is a diffeomorphism which preserves the canonical symplectic two-form ω. A symplectic chart is a chart such that the local expression of ω is the standard symplectic form of the local model space $\mathbb{R}^{2n}$. The symplectic form on $\mathbb{P}(\mathscr{H})$ comes from the Kähler metric $\mathscr{K}$; it is therefore natural to consider Kähler isomorphisms, that is, those diffeomorphisms which preserve the Kähler metric.

The case of the projective space $\mathbb{P}(\mathscr{H})$ parallels closely the case of the n-dimensional projective space $\mathbb{P}(\mathbb{C}^{n+1})$ (for the properties of $\mathbb{P}(\mathbb{C}^{n+1})$ see Chern (1979)). Actually, as in the finite-dimensional case, the standard Kähler metric on the local model space is the projection of the metric given by the scalar product of $\mathscr{H}$. A chart such that the local expression of $\mathscr{K}$ is the standard one is said to be Kählerian. The atlas $\mathscr{A}$ defined in § 2 is therefore Kählerian (i.e. it consists of Kählerian charts), while the atlas $\bar{\mathscr{A}}$ is not, since the local expression of $\mathscr{K}$ is the conjugate of the standard one; we call such an atlas anti-Kählerian.

Kähler isomorphisms are equivalently characterised as those diffeomorphisms which bring Kählerian charts in Kählerian charts, that is, which do not affect the standard local expression of $\mathscr{K}$. As in the case of $\mathbb{P}(\mathbb{C}^{n+1})$, the invariance group of $\mathscr{K}$ is $\mathscr{U}(\mathscr{H})/\sim$. Actually, a unitary operator U brings Kählerian charts in Kählerian charts, and therefore induces a Kähler isomorphism. Moreover, each Kähler isomorphism is induced in such a way by a unitary operator, which is unique up to a phase factor. Indeed, let φ be a Kähler isomorphism of $\mathbb{P}(\mathscr{H})$, $h, h' \in \mathscr{H}$, $\|h\| = \|h'\| = 1$, such that $\varphi([h]) = [h']$. The map $\varphi_{h'} \circ \varphi \circ \varphi_h^{-1}$ is a unitary operator $U_{hh'}: \{h\}^\perp \to \{h'\}^\perp$, which extends uniquely to a unitary operator $U: \mathscr{H} = \{h\}^\perp \oplus \mathbb{C}h \to \mathscr{H} = \{h'\}^\perp \oplus \mathbb{C}h'$, such that $Uh = h'$. U implements φ as required, and a different choice of h, h' changes U by a phase factor. We have therefore a bijection between the automorphism group of $\mathscr{B}(\mathscr{H})$ and the group of Kähler isomorphisms of $\mathbb{P}(\mathscr{H})$. For instance, under an automorphism of $\mathscr{B}(\mathscr{H})$ the energy function (6) becomes $\mathscr{k}_{H'} = (\psi|U^+HU\psi) \cdot (\psi|\psi)^{-1}$, while under the corresponding Kähler isomorphism, it becomes $\mathscr{k}'_H = (U\psi|HU\psi) \cdot (U\psi|U\psi)^{-1} = \mathscr{k}_{H'}$.

Let us consider again the anti-Kählerian atlas $\bar{\mathscr{A}}$. Kähler anti-isomorphisms of $\mathbb{P}(\mathscr{H})$ (which brings Kählerian charts in anti-Kählerian charts) are in bijection with the group $\overline{\mathscr{U}(\mathscr{H})}/\sim$, where $\overline{\mathscr{U}(\mathscr{H})}$ is the group of anti-unitary operators on $\mathscr{H}$, that is, the group $\mathscr{U}(\bar{\mathscr{H}})$ of unitary operators on $\bar{\mathscr{H}}$. The group $\overline{\mathscr{U}(\mathscr{H})}/\sim \simeq \mathscr{U}(\bar{\mathscr{H}})/\sim$ is the group of automorphisms of $\overline{\mathscr{B}(\mathscr{H})} \simeq \mathscr{B}(\bar{\mathscr{H}})$.

We are now ready to give a reformulation, from the geometrical point of view, of Wigner's theorem. For an algebraic reformulation, see Schultz (1982).

Wigner's theorem in its classical form says that a bijection of $\mathbb{P}(\mathscr{H})$ which preserves transition probabilities is implemented either by a unitary or by an anti-unitary operator on $\mathscr{H}$, unique up to a phase factor. But, just up to the phase factor, unitary and anti-unitary operators on $\mathscr{H}$ correspond to Kähler isomorphisms and anti-isomorphisms of $\mathbb{P}(\mathscr{H})$ respectively. A Kähler isomorphism of $\mathbb{P}(\mathscr{H})$ therefore preserves transition probabilities and affine orientation, while a Kähler anti-isomorphism of $\mathbb{P}(\mathscr{H})$ (which corresponds to a Kähler isomorphism of $\mathbb{P}(\bar{\mathscr{H}})$ equipped with its canonical atlas $\mathscr{A}_{(\bar{\mathscr{H}})} \simeq \bar{\mathscr{A}}_{(\mathscr{H})}$) preserves transition probabilities and reverses affine orientation. We remark that only Kähler isomorphisms are holomorphic bijections of $\mathbb{P}(\mathscr{H})$ with the holomorphic structure given by $\mathscr{A}_{(\mathscr{H})}$.

We can therefore restate Wigner's theorem as follows:

Theorem 3 (Wigner). A bijection of $\mathbb{P}(\mathscr{H})$ which preserves transition probabilities is either a Kähler isomorphism or anti-isomorphism of $\mathbb{P}(\mathscr{H})$.

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